CS 2500: Algorithms
Missouri University of Science and Technology
Midterm Exam
Fall 2024

(***Maximum Points: 50***)
(***Time allowed: 65 minutes***)
*(Candidates are allowed additional 10 minutes for **only** reading the paper. They must NOT start writing during this time.)*

This Paper consists of 5 printed pages and 1 blank page.

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

*Answer **five** questions from Part I and **three** questions from Part II.*

Questions are numbered using Arabic numerals $(1, 2, 3, \dots)$*. If the question contains subparts, then you are required to answer **every subpart** of that question. If you have any doubts, please ask the proctor for clarification.*

The intended points for questions or parts of questions are given in brackets [].

You are required to clearly show all steps involved in solving each problem. This includes clearly stating any assumptions made during the problem-solving process, providing step-by-step calculations or logical deductions, and demonstrating how you arrive at your final answer. Each step should be justified with appropriate reasoning, explanations, or theorems where applicable.

Failure to provide clear workings, explanations, or logical reasoning may result in a deduction of points, even if the final answer is correct.

Plan your time wisely. Do not spend too much time on any one problem. Read through all of them first and attack them in the order that allows you to make the most progress.

Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.

Important: *Please ensure that you write (1) full name, (2) student ID, and (3) page number on **EVERY** answer sheet provided. Sheets may get separated during grading.*

ALL THE BEST!

PART I [20 Points]

*Answer **any five** questions.*

1. Consider the following three functions:

$$f_1(n) = 10^n \quad f_2(n) = n^{\log n} \quad f_3(n) = n^{\sqrt{n}}$$

Which of the following options arranges the functions in the increasing order of asymptotic growth rate? [4]

(a) f_3, f_2, f_1

(b) f_2, f_1, f_3

(c) f_1, f_2, f_3

(d) f_2, f_3, f_1

2. The recurrence relation capturing the optimal time of the Tower of Hanoi problem with n discs is: [4]

(a) $T(n) = 2T(n-2) + 2$

(b) $T(n) = 2T(n-1) + n$

(c) $T(n) = 2T(n/2) + 1$

(d) $T(n) = 2T(n-1) + 1$

3. Let H be a binary min-heap consisting of n elements implemented as an array. What is the worst case time complexity of the optimal algorithm to find the maximum element in H? [4]

(a) $\Theta(1)$

(b) $\Theta(\log n)$

(c) $\Theta(n)$

(d) $\Theta(n \log n)$

4. The worst case running times of insertion sort, merge sort, and quick sort, respectively, are: [4]

(a) $\Theta(n \log n)$, $\Theta(n \log n)$, $\Theta(n^2)$

(b) $\Theta(n^2)$, $\Theta(n^2)$, $\Theta(n \log n)$

(c) $\Theta(n^2)$, $\Theta(n \log n)$, $\Theta(n \log n)$

(d) $\Theta(n^2)$, $\Theta(n \log n)$, $\Theta(n^2)$

5. Let $f(n) = n$ and $g(n) = n^{(1+\sin n)}$, where n is a positive integer. Which of the following statements is/are correct? [4]

I. $f(n) = O(g(n))$
II. $f(n) = \Omega(g(n))$

(a) Only I.
(b) Only II.
(c) Both I and II.
(d) Neither I nor II.

6. Which of the following correctly determines the solution of the recurrence equation below: [4]

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2T\left(\frac{n}{2}\right) + n & \text{if } n > 1. \end{cases}$$

(a) $\Theta(n)$
(b) $\Theta(n \log n)$
(c) $\Theta(n^2)$
(d) $\Theta(\log n)$

7. You have an array with n elements. Suppose you implement quick sort by always choosing the middle element of the array as the pivot. Then, the tightest upper bound for the worst case performance is: [4]

(a) $O(n^2)$
(b) $O(n \log n)$
(c) $\Theta(n \log n)$
(d) $O(n^2)$

PART II [30 Points]

*Answer **any three** questions.*

1. (a) Design an algorithm to reverse the digits of a given number. Express your algorithm in pseudo-code and write the time complexity in Big-O notation. [4]

(b) A logistics company is analyzing the total fuel consumption required for delivering packages along a series of routes. The fuel consumption for the i-th route is proportional to i^k liters, where $k \geq 0$ is a constant that reflects the increasing difficulty of each route. The total fuel consumption after completing n routes is given by the function:

$$F(n) = \sum_{i=1}^{n} i^k$$

Prove that $F(n) = O(n^{k+1})$ for $k \geq 0$. [6]

2. (a) Let us assume that the analysis of an algorithm yields $f(n)$ as follows:

$$f(n) = \sum_{i=1}^{n} 6i(i+1)$$

Determine the total number of operations and provide the time complexity of the algorithm in Big-O notation. [5]

(b) Consider the following recurrence relation:

$$T(n) = 3T\left(\frac{n}{2}\right) + n$$

Solve the recurrence using the Recurrence Tree Method. You must show the recurrence tree and calculations at each level. What is the time complexity of $T(n)$ in Big-O notation? [5]

3. (a) Let

$$f(n) = \frac{2n^3 + 13\log_2 n}{7n^2}$$

for an algorithm. Prove that $f(n)$ is in $O(n)$ [4]

(b) The recurrence equation

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 3T(n \div 2) + n & \text{otherwise} \end{cases}$$

has a solution $T(2^k) = 3^{k+1} - 2^{k+1}$, when $n = 2^k$ and $n > 1$. Use mathematical induction to verify that the solution $T(2^k) = 3^{k+1} - 2^{k+1}$ holds for all k. [6]

4. In an image processing system, a grayscale image is represented by a singly linked list, where each node stores the intensity value of a pixel (ranging from 0 to 255). The linked list represents a single row of pixel data in the image.

(a) Design an algorithm that swaps every two adjacent pixel values in the linked list. You should write the algorithm in pseudocode. [3]

(b) Derive the recurrence relation for the time complexity of the algorithm you designed, assuming the linked list contains n pixels. [4]

(c) Solve the recurrence relation and write the overall time complexity of the algorithm. [3]

5. In a job scheduling system, tasks are represented by their execution times. We want to organize the tasks so that the most time-consuming tasks are executed first. The array

$$A = [5, 13, 2, 25, 7, 17, 20, 8, 4]$$

represents the execution times of 9 tasks.

(a) Illustrate the operation of `Build-Max-Heap` on the array A. Show the array at each step of the heap-building process. [6]

(b) How would your approach change if we wanted to organize the tasks so that the least time-consuming tasks are executed first? [4]