CS 2500: Algorithms

Lecture 5: Mathematical Induction

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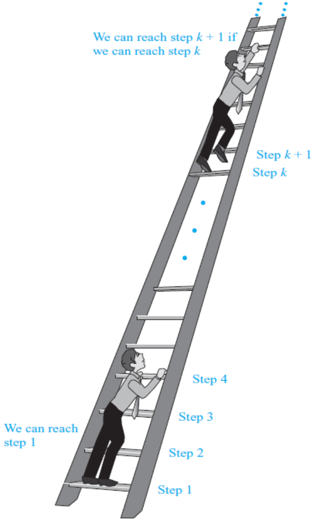
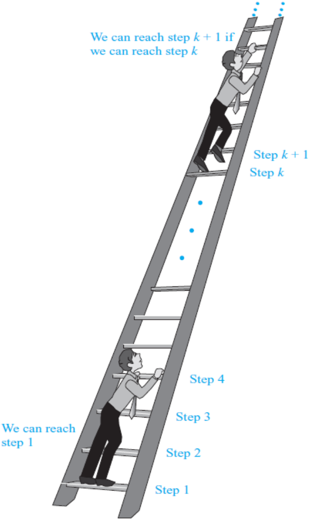
Mathematical Induction: Infinite Ladder Example

- Suppose we have an infinite ladder.
- We want to know whether we can reach every step on this ladder.
- We know two things:
  1. We can reach the first rung of the ladder.
  2. If we can reach a particular rung, we can reach the next rung.
- **Can we conclude that we can reach every rung?**



Mathematical Induction: Infinite Ladder Example

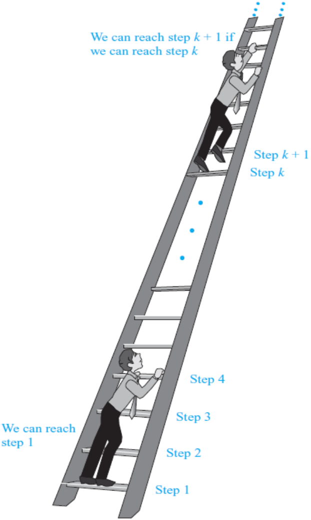
- By (1), we can reach the first rung of the ladder.
- Since we can reach the first rung, by (2), we can reach the second rung.
- Continuing in this way:
  - We can reach the third rung, the fourth rung, and so on.
  - For example, after 100 uses of (2), we know that we can reach the 101st rung.



Mathematical Induction: Infinite Ladder Example

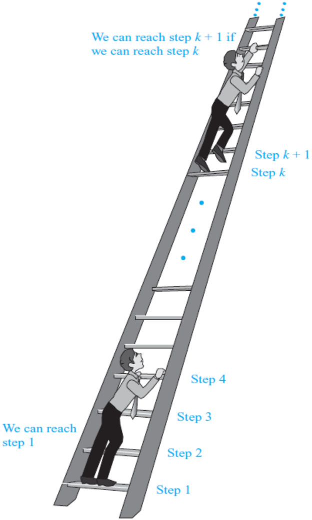
Can We Reach Every Rung?

- Yes, we can reach every rung of the infinite ladder.
- This can be verified using an important proof technique called **mathematical induction**.
- We can show that $P(n)$ is true for every positive integer n.
- Where $P(n)$ is the statement that we can reach the nth rung of the ladder.



Importance of Mathematical Induction

- Mathematical induction is a crucial proof technique for assertions like this.
- It is extensively used to prove results about:
  - Complexity of algorithms.
  - Correctness of computer programs.
  - Theorems about graphs and trees.
  - Various identities and inequalities.



Mathematical Induction: General Concept

- Mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n.
- $P(n)$ is a propositional function.
  - Propositional function: a statement or expression that contains one or more variables and becomes a proposition when the variables are replaced by specific values.
  - Example: $P(x)$: x is an even number.
    - If $x = 1$, then $P(x)$ is true.
    - If $x = 3$, then $P(x)$ is false.

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, we complete two steps:

- **Basis Step:** We verify that $P(1)$ is true.
- **Inductive Step:** Show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

Inductive Step in Detail

- To complete the inductive step, assume that $P(k)$ is true for an arbitrary positive integer k.
- Show that under this assumption, $P(k+1)$ must also be true.
- The assumption that $P(k)$ is true is called the **inductive hypothesis**.

- Once both steps are complete, we have shown that $P(n)$ is true for all positive integers n.
- This means $\forall n, P(n)$ is true where the quantification is over the set of positive integers.

Rule of Inference for Mathematical Induction

Expressed as a rule of inference, this proof technique can be stated as:

$$(P(1) \wedge \forall k(P(k) \to P(k+1))) \to \forall n\, P(n)$$

when the domain is the set of positive integers.

- Mathematical induction is a critical proof technique used in many areas of mathematics and computer science.

Remark on Mathematical Induction

- In a proof by mathematical induction, it is **not assumed** that $P(k)$ is true for all positive integers.
- It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true.
- Thus, a proof by mathematical induction is not a case of begging the question or circular reasoning.

Why Mathematical Induction is Valid

- The validity of mathematical induction comes from the well-ordering property of the set of positive integers.
- **Well-Ordering Property:** Every nonempty subset of the set of positive integers has a least element.

Example:

- Consider the subset $S = \{n \in \mathbb{Z}^+ \mid n \geq 3\}$.
- Explicitly, $S = \{3, 4, 5, 6, 7, \ldots\}$.
- According to the well-ordering property, S must have a least element.
- The least element in this subset S is 3.

Why Mathematical Induction is Valid

- Suppose $P(1)$ is true, and the proposition $P(k) \to P(k+1)$ is true for all positive integers k.
- To show $P(n)$ is true for all positive integers n, assume there is at least one positive integer for which $P(n)$ is false.

Why Assume $P(n)$ is False?

- The assumption that $P(n)$ is false for some n is a technique in **proof by contradiction**.
- Goal: Prove that $P(n)$ is true for all positive integers n.
- Strategy: Assume the opposite of what you want to prove.

Steps of the Contradiction

- **Step 1: Assume the Opposite**
  - Assume there is at least one positive integer n for which $P(n)$ is false.
  - Define S as the set of all positive integers where $P(n)$ is false:

$$S = \{n \in \mathbb{Z}^+ \mid P(n) \text{ is false}\}$$

  - Because we assumed that $P(n)$ is false for some positive integer n, the set S must contain at least one element.
  - Therefore, S is non-empty.
- **Step 2: Well-Ordering Property**
  - The set S has a least element, say m.
  - m is the smallest positive integer for which $P(m)$ is false.

Analyzing the Situation

- m cannot be 1 because $P(1)$ is true by the basis step.
- Since m is greater than 1, $m - 1$ must be a positive integer.
- $P(m - 1)$ must be true because m is the smallest element in S.

Inductive Hypothesis

- Given that $P(m-1)$ is true, and using the inductive step $P(k) \rightarrow P(k+1)$:
- We conclude that $P(m)$ must also be true.

- **Contradiction:** This conclusion contradicts the assumption that $P(m)$ is false.

Conclusion

- Since the assumption leads to a contradiction, the original assumption must be false.
- Therefore, $P(n)$ must be true for all positive integers n.

Example 1: Sum of the First n Positive Integers

Statement of the Problem:

- Prove that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

- Let $P(n)$ be the proposition that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.
- We will prove $P(n)$ using mathematical induction.

Example 1: Sum of the First n Positive Integers

Steps of the Induction Proof

- Show that $P(1)$ is true (Basis Step).
- Show that $P(k)$ implies $P(k+1)$ for all $k \geq 1$ (Inductive Step).

Basis Step

- $P(1)$ is the statement:

$$1 = \frac{1(1+1)}{2}$$

- The left-hand side is 1 (the sum of the first positive integer).
- The right-hand side is also 1:

$$\frac{1(2)}{2} = \frac{2}{2} = 1.$$

- Thus, $P(1)$ is true.

Example 1: Sum of the First n Positive Integers

Inductive Step: Statement

- Assume that $P(k)$ is true for some arbitrary positive integer k.
- That is, assume:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

- We must show that $P(k+1)$ is true, which means proving:

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Inductive Step: Proof

- Starting with the inductive hypothesis:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2},$$

- Add $(k+1)$ to both sides:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

Simplifying the Expression

- Factor $(k+1)$ out of the right-hand side:

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2}.$$

- Simplify the expression:

$$\frac{(k+1)(k+2)}{2}.$$

- This shows that $P(k+1)$ is true.

Example 1: Sum of the First n Positive Integers

Conclusion

- We have shown that $P(1)$ is true (Basis Step).
- We have shown that $P(k)$ implies $P(k+1)$ (Inductive Step).
- By mathematical induction, $P(n)$ is true for all positive integers n.
- Therefore, we have proven that:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers n.

Example 2: Sum of the First n Positive Odd Integers

Problem Statement

- Conjecture a formula for the sum of the first n positive odd integers.
- The sums for $n = 1, 2, 3, 4, 5$ are:

$$1 = 1, \quad 1 + 3 = 4, \quad 1 + 3 + 5 = 9,$$

$$1 + 3 + 5 + 7 = 16, \quad 1 + 3 + 5 + 7 + 9 = 25.$$

- Conjecture: The sum of the first n positive odd integers is n^2.
- Formula: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Example 2: Sum of the First n Positive Odd Integers

Steps of the Induction Proof

- Let $P(n)$ denote the proposition that the sum of the first n positive odd integers is n^2.
- We will prove $P(n)$ using mathematical induction:
  1. Basis Step: Show that $P(1)$ is true.
  2. Inductive Step: Show that $P(k) \to P(k+1)$ is true for all $k \geq 1$.

Example 2: Sum of the First n Positive Odd Integers

Basis Step

- $P(1)$ states that the sum of the first one positive odd integer is $1^2 = 1$.
- This is true because the sum is indeed 1.
- Therefore, $P(1)$ is true.

Example 2: Sum of the First n Positive Odd Integers

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary positive integer k.
- That is, assume:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

- We must show that $P(k + 1)$ is true:

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

Example 2: Sum of the First n Positive Odd Integers

Inductive Step: Proof

- Starting with the inductive hypothesis:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2,$$

- Add $(2k + 1)$ to both sides:

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1).$$

Simplifying the Expression

- Simplify the right-hand side:

$$k^2 + 2k + 1 = (k + 1)^2.$$

- This shows that $P(k + 1)$ follows from $P(k)$.
- Therefore, $P(k + 1)$ is true.

Example 2: Sum of the First n Positive Odd Integers

Conclusion

- We have shown that $P(1)$ is true (Basis Step).
- We have shown that $P(k) \rightarrow P(k+1)$ (Inductive Step).
- By mathematical induction, $P(n)$ is true for all positive integers n.
- Therefore, we have proven that:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

  for all positive integers n.

Example 3: Sum of Powers of 2

Problem Statement

- Use mathematical induction to prove the following formula:

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

  for all nonnegative integers n.

- Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Steps of the Induction Proof

- We will prove $P(n)$ using mathematical induction:
  1. Basis Step: Show that $P(0)$ is true.
  2. Inductive Step: Show that $P(k) \to P(k+1)$ is true for all $k \geq 0$.

Example 3: Sum of Powers of 2

Basis Step

- $P(0)$ states that:

$$1 = 2^{0+1} - 1 = 2^1 - 1 = 1.$$

- This is true, so the basis step is complete.

Example 3: Sum of Powers of 2

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary nonnegative integer k.
- That is, assume:

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

- We must show that $P(k+1)$ is true:

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1.$$

Example 3: Sum of Powers of 2

Inductive Step: Proof

- Start with the inductive hypothesis:

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

- Add 2^{k+1} to both sides:

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1}.$$

Example 3: Sum of Powers of 2

Simplifying the Expression

- Simplify the right-hand side:

$$(2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1.$$

- This shows that $P(k+1)$ follows from $P(k)$.
- Therefore, $P(k+1)$ is true.

Example 3: Sum of Powers of 2

Conclusion

- We have shown that $P(0)$ is true (Basis Step).
- We have shown that $P(k) \to P(k+1)$ (Inductive Step).
- By mathematical induction, $P(n)$ is true for all nonnegative integers n.
- Therefore, we have proven that:

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Example 4: Proving $n < 2^n$ for All Positive Integers n

Problem Statement

- Use mathematical induction to prove the inequality:

$$n < 2^n$$

  for all positive integers n.
- Let $P(n)$ be the proposition that $n < 2^n$.

Example 4: Proving $n < 2^n$ for All Positive Integers n

Steps of the Induction Proof

- We will prove $P(n)$ using mathematical induction:
  1. Basis Step: Show that $P(1)$ is true.
  2. Inductive Step: Show that $P(k) \to P(k+1)$ is true for all $k \geq 1$.

Example 4: Proving $n < 2^n$ for All Positive Integers n

Basis Step

- $P(1)$ states that:
$$1 < 2^1 = 2.$$
- This is true, so the basis step is complete.

Example 4: Proving $n < 2^n$ for All Positive Integers n

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary positive integer k.
- That is, assume:

$$k < 2^k.$$

- We must show that $P(k+1)$ is true:

$$k + 1 < 2^{k+1}.$$

Example 4: Proving $n < 2^n$ for All Positive Integers n

Inductive Step: Proof

- Start with the inductive hypothesis:

$$k < 2^k.$$

- Add 1 to both sides:

$$k + 1 < 2^k + 1.$$

- Note that $1 \leq 2^k$, so:

$$2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Example 4: Proving $n < 2^n$ for All Positive Integers n

Conclusion

- Therefore, $k + 1 < 2^{k+1}$, which shows that $P(k + 1)$ is true.
- The induction step is complete.
- By mathematical induction, $P(n)$ is true for all positive integers n.
- Therefore, we have proven that:

$$n < 2^n$$

for all positive integers n.

Example 5: Proving $2^n < n!$ for All Integers $n \geq 4$

Problem Statement

- Use mathematical induction to prove that:

$$2^n < n!$$

  for every integer n with $n \geq 4$.
- Let $P(n)$ be the proposition that $2^n < n!$.

Example 5: Proving $2^n < n!$ for All Integers $n \geq 4$

Basis Step

- Since the inequality is false for $n = 1, 2, 3$, we begin with $P(4)$.
- $P(4)$ states that:
$$2^4 = 16 < 24 = 4!$$
- This is true, so the basis step is complete.

Example 5: Proving $2^n < n!$ for All Integers $n \geq 4$

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary integer $k \geq 4$.
- That is, assume:

$$2^k < k!.$$

- We must show that $P(k+1)$ is true:

$$2^{k+1} < (k+1)!.$$

Inductive Step: Proof

- Start with the definition of exponent:

$$2^{k+1} = 2 \cdot 2^k.$$

- Apply the inductive hypothesis:

$$2^{k+1} < 2 \cdot k!.$$

- Since $2 < k + 1$ for $k \geq 4$:

$$2 \cdot k! < (k+1) \cdot k! = (k+1)!.$$

Example 5: Proving $2^n < n!$ for All Integers $n \geq 4$

Conclusion

- This shows that $P(k+1)$ is true when $P(k)$ is true.
- The induction step is complete.
- By mathematical induction, $P(n)$ is true for all integers $n \geq 4$.
- Therefore, we have proven that:

$$2^n < n!$$

for all integers $n \geq 4$.

Example 6: Proving $n^3 - n$ is Divisible by 3

Problem Statement

- Use mathematical induction to prove that:

$$n^3 - n \text{ is divisible by } 3$$

  for all positive integers n.
- Let $P(n)$ be the proposition that $n^3 - n$ is divisible by 3.

Example 6: Proving $n^3 - n$ is Divisible by 3

Basis Step

- $P(1)$ states that:

$$1^3 - 1 = 0,$$

  which is divisible by 3.
- Therefore, $P(1)$ is true, completing the basis step.

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary positive integer k.
- That is, assume:

$$k^3 - k \text{ is divisible by 3.}$$

- We must show that $P(k+1)$ is true:

$$(k+1)^3 - (k+1) \text{ is divisible by 3.}$$

Example 6: Proving $n^3 - n$ is Divisible by 3

Inductive Step: Proof

- Expand $(k+1)^3 - (k+1)$:

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1).$$

- Simplify the expression:

$$= (k^3 - k) + 3(k^2 + k).$$

Example 6: Proving $n^3 - n$ is Divisible by 3

Conclusion

- By the inductive hypothesis, $k^3 - k$ is divisible by 3.
- The second term, $3(k^2 + k)$, is clearly divisible by 3.
- Therefore, $(k + 1)^3 - (k + 1)$ is divisible by 3, completing the inductive step.
- By mathematical induction, $P(n)$ is true for all positive integers n.

Example 7: Proving $7^{n+2} + 8^{2n+1}$ is Divisible by 57

Problem Statement

- Use mathematical induction to prove that:

$$7^{n+2} + 8^{2n+1} \text{ is divisible by } 57$$

  for all nonnegative integers n.
- Let $P(n)$ be the proposition that $7^{n+2} + 8^{2n+1}$ is divisible by 57.

Example 7: Proving $7^{n+2} + 8^{2n+1}$ is Divisible by 57

Basis Step

- $P(0)$ states that:

$$7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 49 + 8 = 57,$$

  which is divisible by 57.
- Therefore, $P(0)$ is true, completing the basis step.

Example 7: Proving $7^{n+2} + 8^{2n+1}$ is Divisible by 57

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary nonnegative integer k.
- That is, assume:

$$7^{k+2} + 8^{2k+1} \text{ is divisible by 57.}$$

- We must show that $P(k+1)$ is true:

$$7^{(k+1)+2} + 8^{2(k+1)+1} = 7^{k+3} + 8^{2k+3} \text{ is divisible by 57.}$$

Inductive Step: Proof

- Start with:

$$7^{k+3} + 8^{2k+3} = 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1}.$$

- Recognize that:

$$8^2 = 64 = 57 + 7.$$

- Substituting:

$$7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} = 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.$$

Conclusion

- By the inductive hypothesis, $7^{k+2} + 8^{2k+1}$ is divisible by 57.
- The second term, $57 \cdot 8^{2k+1}$, is clearly divisible by 57.
- Therefore, $7^{k+3} + 8^{2k+3}$ is divisible by 57, completing the inductive step.
- By mathematical induction, $P(n)$ is true for all nonnegative integers n.

Example 8: Proving a Set with n Elements has 2^n Subsets

Problem Statement

- Use mathematical induction to prove that a set with n elements has 2^n subsets.
- Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

Example 8: Proving a Set with n Elements has 2^n Subsets

Basis Step

- $P(0)$ is true because a set with zero elements, the empty set, has exactly $2^0 = 1$ subset (itself).
- Therefore, $P(0)$ is true, completing the basis step.

Example 8: Proving a Set with n Elements has 2^n Subsets

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary non-negative integer k.
- That is, assume:

  A set with k elements has 2^k subsets.

- We must show that $P(k+1)$ is true:

  A set with $k+1$ elements has 2^{k+1} subsets.

Example 8: Proving a Set with n Elements has 2^n Subsets

Inductive Step: Proof

- Let T be a set with $k+1$ elements.
- We can write $T = S \cup \{a\}$, where S is a set with k elements.
- Each subset X of S can be expanded to two subsets of T: X and $X \cup \{a\}$.
- Therefore, there are $2 \times 2^k = 2^{k+1}$ subsets of T.

Example 8: Proving a Set with n Elements has 2^n Subsets

Conclusion

- This shows that $P(k+1)$ is true when $P(k)$ is true.
- The induction step is complete.
- By mathematical induction, $P(n)$ is true for all nonnegative integers n.
- Therefore, we have proven that a set with n elements has 2^n subsets.

Example 9: Generalization of De Morgan's Law

Problem Statement

- Use mathematical induction to prove the following generalization of De Morgan's law:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

  whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

- Let $P(n)$ be the identity for n sets.

Example 9: Generalization of De Morgan's Law

Basis Step

- The statement $P(2)$ asserts that:

$$\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}.$$

- This is one of De Morgan's laws which we know to be true.
- Therefore, $P(2)$ is true, completing the basis step.

Example 9: Generalization of De Morgan's Law

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary integer $k \geq 2$.
- That is, assume:

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}.$$

- We must show that $P(k+1)$ is true:

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j}.$$

Example 9: Generalization of De Morgan's Law

Inductive Step: Proof

$$
\begin{aligned}
\overline{\bigcap_{j=1}^{k+1} A_j} &= \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}} && \text{by the definition of intersection} \\
&= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} && \text{by De Morgan's law (where the two sets are } \textstyle\bigcap_{j=1}^{k} A_j \text{ and } A_{k+1}\text{)} \\
&= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} && \text{by the inductive hypothesis} \\
&= \bigcup_{j=1}^{k+1} \overline{A_j} && \text{by the definition of union.}
\end{aligned}
$$

Conclusion

- This shows that $P(k+1)$ is true when $P(k)$ is true.
- The induction step is complete.
- By mathematical induction, $P(n)$ is true for all integers $n \geq 2$.
- Therefore, we have proven that:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

for all $n \geq 2$.

Example 10: Proving the Set Identity

Problem Statement

- Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then:

  $$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

- Let $P(n)$ denote this identity for n sets $A_1, A_2, \ldots, A_n$.

Example 10: Proving the Set Identity

Basis Step

- For $n = 1$, the statement is:

$$(A_1) \cup B = (A_1 \cup B).$$

- This is trivially true, as both sides are the same.
- Therefore, $P(1)$ is true, completing the basis step.

Example 10: Proving the Set Identity

Inductive Step: Statement

- Assume $P(k)$ is true for some arbitrary positive integer k.
- That is, assume:

$$(A_1 \cap A_2 \cap \cdots \cap A_k) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B).$$

- We must show that $P(k+1)$ is true:

$$(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_{k+1} \cup B).$$

Example 10: Proving the Set Identity

Inductive Step: Proof

- Start with the left-hand side of $P(k+1)$:

$$(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) \cup B.$$

- By associative and distributive properties of sets:

$$= [(A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}] \cup B.$$

- Apply the distributive law:

$$= [(A_1 \cap A_2 \cap \cdots \cap A_k) \cup B] \cap (A_{k+1} \cup B).$$

- Apply the inductive hypothesis:

$$= [(A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B)] \cap (A_{k+1} \cup B).$$

Example 10: Proving the Set Identity

Conclusion

- This simplifies to:

$$= (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_{k+1} \cup B).$$

- This shows that $P(k+1)$ is true when $P(k)$ is true.
- The induction step is complete.
- By mathematical induction, $P(n)$ is true for all positive integers n.
- Therefore, we have proven the set identity for any number of sets $A_1, A_2, \ldots, A_n$ and set B.

Mistaken Proof by Mathematical Induction

Problem Statement

- Find the error in this "proof" of the clearly false claim: "Every set of lines in the plane, no two of which are parallel, meet in a common point."
- Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point.

Mistaken Proof by Mathematical Induction

Basis Step

- The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point (by the definition of parallel lines).
- Therefore, the basis step is correctly completed.

Mistaken Proof by Mathematical Induction

Inductive Step: Statement

- Assume $P(k)$ is true for some positive integer k.
- That is, assume every set of k lines in the plane, no two of which are parallel, meet in a common point.
- We must show that $P(k+1)$ is true:
- That is, every set of $k+1$ lines in the plane, no two of which are parallel, meet in a common point.

Mistaken Proof by Mathematical Induction

Inductive Step: Proof

- Consider a set of $k+1$ distinct lines in the plane.
- By the inductive hypothesis, the first k of these lines meet in a common point p_1.
- By the inductive hypothesis, the last k of these lines meet in a common point p_2.
- We show that p_1 and p_2 must be the same point:
  - If p_1 and p_2 were different, all lines containing both must be the same line (since two points determine a line).
  - This contradicts the assumption that all lines are distinct.
  - Therefore, $p_1 = p_2$.
- This supposedly completes the inductive step.

Identifying the Error

- Examining this proof by mathematical induction, it appears that everything is in order.
- However, there is a subtle error:
    - The inductive step requires $k \geq 3$.
    - When $k = 2$, the goal is to show that every three distinct lines meet in a common point.
    - The first two lines meet in a common point p_1, and the last two meet in a common point p_2.
    - In this case, p_1 and p_2 do not have to be the same because only the second line is common to both sets.
- This is where the inductive step fails.

Mistaken Proof by Mathematical Induction

Conclusion

- The supposed proof by mathematical induction is incorrect.
- The error lies in the inductive step, specifically when trying to extend the result from $k = 2$ to $k + 1 = 3$.
- This example highlights the importance of carefully verifying each step in a proof by mathematical induction.

Guidelines for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.

Guidelines for Proofs by Mathematical Induction

6. Prove the statement $P(k+1)$ making use of the assumption $P(k)$. Be sure that your proof is valid for all integers $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers $n \geq b$.